

Tensorial representations of Reynolds-stress pressure-strain redistribution

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*The purpose of the present note is to contribute in clarifying the relation between representation bases used in the closure for the redistribution (pressure-strain) tensor ϕ_{ij} , and to construct representation bases whose elements have clear physical significance. The representation of different models in the same basis is essential for comparison purposes, and the definition of the basis by physically meaningful tensors adds insight to our understanding of closures. The rate-of-production tensor can be split into production by mean strain and production by mean rotation $P_{ij} = P_{\bar{S}_{ij}} + P_{\bar{\Omega}_{ij}}$. The classic representation basis $\mathfrak{B}[\mathbf{b}, \bar{\mathbf{S}}, \bar{\mathbf{\Omega}}]$ of homogeneous turbulence [eg Ristorcelli J.R., Lumley J.L., Abid R.: *J. Fluid Mech.* **292** (1995) 111–152], constructed from the anisotropy \mathbf{b} , the mean strain-rate $\bar{\mathbf{S}}$, and the mean rotation-rate $\bar{\mathbf{\Omega}}$ tensors, is interpreted, in the present work, in terms of the relative contributions of the deviatoric tensors $P_{\bar{S}_{ij}}^{(\text{dev})} := P_{\bar{S}_{ij}} - \frac{2}{3}P_k\delta_{ij}$ and $P_{\bar{\Omega}_{ij}}^{(\text{dev})} := P_{\bar{\Omega}_{ij}}$. Different alternative equivalent representation bases, explicitly using $P_{\bar{S}_{ij}}^{(\text{dev})}$ and $P_{\bar{\Omega}_{ij}}$ are discussed, and the projection rules between bases are calculated, using a matrix-based systematic procedure. An initial term-by-term a priori investigation of different second-moment closures is undertaken.*

1 Introduction

The pressure-strain (redistribution) correlation [1, 2, 3]

$$\phi_{ij} := 2p'\bar{S}'_{ij} = \phi_{ij}^{(\text{r};\mathfrak{T})} + \phi_{ij}^{(\text{s};\mathfrak{T})} + \phi_{ij}^{(\text{w})} \quad (1)$$

“plays a pivotal role in determining the structure of a wide variety of turbulent flows” [4]. In (1), p is the pressure, $S_{ij} := \frac{1}{2}(\partial_{x_j}u_i + \partial_{x_i}u_j)$ is the rate-of-strain tensor, u_i are the velocity-components in the reference-frame with coordinates x_i , $(\bar{\cdot})$ denotes ensemble averaging, $(\bar{\cdot})'$ denotes turbulent fluctuations, $\phi_{ij}^{(\text{r};\mathfrak{T})}$ is the rapid (meanflow/turbulence interaction) volume-part of redistribution [4], $\phi_{ij}^{(\text{s};\mathfrak{T})}$ is the slow (turbulence/turbulence interaction) volume-part of redistribution [4], and $\phi_{ij}^{(\text{w})} := \phi_{ij}^{(\text{r};\mathfrak{T})} + \phi_{ij}^{(\text{s};\mathfrak{T})}$ are the wall-echo

terms near solid boundaries [5], obtained by the free-space Green’s function solution of the incompressible flow Poisson equation (linear in p') for the fluctuating pressure [5]. The most general approach to modelling ϕ_{ij} in homogeneous turbulence is based on the order-4 tensors associated with the volume integrals $\phi_{ij}^{(\text{r};\mathfrak{T})}$ and $\phi_{ij}^{(\text{s};\mathfrak{T})}$ [6], which require in particular, through obvious scaling arguments, that retained models for $\phi_{ij}^{(\text{r})}$ should be linear in meanflow-velocity gradients $\partial_{x_j}\bar{u}_i$ [3], and that $\phi_{ij}^{(\text{s})}$ should not depend directly on $\partial_{x_j}\bar{u}_i$ [3]. The redistribution tensor is symmetric by definition (1) and deviatoric, because of the incompressible fluctuating continuity equation ($\phi_{\ell\ell} \stackrel{(1)}{=} 2p'\partial_{x_\ell}\bar{u}_\ell = 0$). Therefore second-moment closures (SMCs) represent redistribution as a linear combination of deviatoric symmetric tensors [4, 6, 7]. Let

$$\bar{S}_{ij} := \frac{1}{2}(\partial_{x_j}\bar{u}_i + \partial_{x_i}\bar{u}_j) \quad (2a)$$

$$\bar{\Omega}_{ij} := \frac{1}{2}(\partial_{x_j}\bar{u}_i - \partial_{x_i}\bar{u}_j) \stackrel{(2a)}{=} \partial_{x_j}\bar{u}_i - \bar{S}_{ij} \quad (2b)$$

$$b_{ij} := (\overline{u'_\ell u'_\ell})^{-1} \overline{u'_i u'_j} - \frac{1}{3}\delta_{ij} \quad (2c)$$

$$P_{\bar{S}_{ij}} := -\rho\overline{u'_i u'_\ell} \bar{S}_{j\ell} - \rho\overline{u'_j u'_\ell} \bar{S}_{i\ell} \quad (2d)$$

$$P_{\bar{\Omega}_{ij}} := -\rho\overline{u'_i u'_\ell} \bar{\Omega}_{j\ell} - \rho\overline{u'_j u'_\ell} \bar{\Omega}_{i\ell} \quad (2e)$$

$$P_{ij} := -\rho\overline{u'_i u'_\ell} \partial_{x_\ell} \bar{u}_j - \rho\overline{u'_j u'_\ell} \partial_{x_\ell} \bar{u}_i \stackrel{(2a-2e)}{=} P_{\bar{S}_{ij}} + P_{\bar{\Omega}_{ij}} \quad (2f)$$

$$D_{ij} := -\rho\overline{u'_i u'_\ell} \partial_{x_j} \bar{u}_\ell - \rho\overline{u'_j u'_\ell} \partial_{x_i} \bar{u}_\ell \stackrel{(2a-2e)}{=} P_{\bar{S}_{ij}} - P_{\bar{\Omega}_{ij}} \quad (2g)$$

viz the Reynolds-stress-anisotropy tensor b_{ij} (2c) [3], the production tensor P_{ij} (2f) appearing in Reynolds-stress transport [4], which can be separated (2f) in production by mean strain-rate $P_{\bar{S}_{ij}}$ (2d) and production by mean rotation-rate $P_{\bar{\Omega}_{ij}}$ (2e). When homogeneous turbulence is submitted to irrotational strain $P_{\bar{\Omega}_{ij}} = 0$, while in case of solid-body rotation $P_{\bar{S}_{ij}} = 0$. The tensor D_{ij} (2g), which bears no particular name, appears in several early pressure-strain closures [2, 8, 9]. Naot et al. [8] attribute to Reynolds [2] its introduction in models for ϕ_{ij} . By (2g), it follows that D_{ij} can also be interpreted, *indeed should be defined*, as the difference of production by strain minus production by rotation.

Almost all practical models¹ for ϕ_{ij} [1, 2, 3, 4, 6, 7, 8, 9, 11, 12] correspond to a linear combination of tensors in (2) and/or tensors constructed by products of tensors in (2) which are linear in the mean-velocity gradients. The particular models range from simple phenomenological representations [1, 11, 8, 2, 9] using only tensors in (2), to more complex models making reference to irreducible representation bases [4, 6, 7]. It is well established that the most general models can be represented in a basis of 8 linearly independent tensors [6]. However, there are several possible choices of the basis-elements, and the originally established forms of different models [1, 2, 3, 4, 6, 7, 8, 9, 11, 12] use different basis-elements. Although most of the models have been projected on a common basis [6], the most complex one [13, 14] has invariably been expressed in a reducible form, including tensors which are linearly dependent [6] and can be projected on an 8-element basis [15]. To make detailed comparisons between different models, going beyond global evaluation of results against data for a given testcase, it is necessary to express the models in a common basis. The purpose of the present work is to contribute in advancing towards the answer to the questions: 1) what is the common representation basis that should be retained, 2) what is the physical significance of the basis-elements, and 3) how different modelling choices associated with different routes followed in the construction of various models can be compared.

In §2 we summarize results concerning the classical 8-element representation basis [6], and debate on the arguments in favour of a polynomial representation basis vs a functional representation basis [4, 6]. In §3 we reinterpret the classical representation basis in terms of production by mean strain-rate $P_{\bar{S}_{ij}}$ (2d) and production by mean rotation-rate $P_{\bar{\Omega}_{ij}}$ (2e), illustrating the physical significance to the 3 basis elements, which represent all quasilinear models for rapid pressure-strain [16]. In §4 we consider alternative bases built from the symmetric tensors $[b_{ij}, P_{ij}, D_{ij}]$ or $[b_{ij}, P_{\bar{S}_{ij}}, P_{\bar{\Omega}_{ij}}]$, which under the requirement of linearity in mean-velocity gradients are symmetric in $[P_{ij}, D_{ij}]$ or $[P_{\bar{S}_{ij}}, P_{\bar{\Omega}_{ij}}]$, respectively, we show that the mean rate-of-strain tensor \bar{S}_{ij} can be explicitly projected on these bases, in which it does not appear explicitly, and discuss the advantages and drawbacks of such symmetric bases generated from tensors appearing in the transport equations for the Reynolds-stresses. In §5 we briefly note how projection of models on different bases can be made systematic using projection matrices and their inverses. Finally in §7, we illustrate, through *a priori* analysis of DNS data for fully developed plane channel flow, how term-by-term comparison of various models, expressed in a common basis, can be used to highlight different modelling strategies, indicating directions for future work (§8).

¹Notice that Launder [10] has suggested to further include the advection tensor $C_{ij} := \rho D_i u_j^* u_j^*$ in the representation, but this has not become standard practice. In the case of spatially evolving stationary quasi-homogeneous turbulence, $C_{ij} = \rho \bar{u}_i \partial_{x_i} u_j^* u_j^* \neq 0$, while in DNS studies of time-evolving spatially homogeneous turbulence, $C_{ij} = \rho \partial_i u_j^* u_j^* \neq 0$, implying that the suggestion of including C_{ij} in the representation merits further study.

2 Classical polynomial representation basis

Recall [17], that every order-2 tensor $\mathbf{A} \in \mathbb{E}^{3 \times 3}$ has 3 invariants, $\mathbf{I}_\mathbf{A} := \text{tr} \mathbf{A}$, $\mathbf{II}_\mathbf{A} := \frac{1}{2}(\text{tr}^2 \mathbf{A} - \text{tr} \mathbf{A}^2) = \frac{1}{2}((A_{\ell\ell})^2 - A_{\ell m} A_{\ell m})$, and $\mathbf{III}_\mathbf{A} := \det \mathbf{A} = \epsilon_{ijk} A_{1i} A_{2j} A_{3k} = \epsilon_{ijk} A_{i1} A_{j2} A_{k3}$, coefficients of its characteristic polynomial $\mathbb{R}[x] \ni p(x; \mathbf{A}) := x^3 - \mathbf{I}_\mathbf{A} x^2 + \mathbf{II}_\mathbf{A} x - \mathbf{III}_\mathbf{A}$, whose roots $\in \mathbb{C}$ (or $\in \mathbb{R}$ if \mathbf{A} is symmetric) are the eigenvalues of \mathbf{A} , and which, by the Cayley-Hamilton theorem, is satisfied by \mathbf{A} , *ie* $\mathbf{A}^3 = \mathbf{I}_\mathbf{A} \mathbf{A}^2 - \mathbf{II}_\mathbf{A} \mathbf{A} + \mathbf{III}_\mathbf{A} \mathbf{I}_3$ [17, (4.24), p. 689]. Applying the Cayley-Hamilton theorem to $(\mathbf{A} + \mathbf{B})^3 - (\mathbf{A} - \mathbf{B})^3$ yields the Cayley-Hamilton theorem extension $\mathbf{ABA} = -\mathbf{A}^2 \mathbf{B} - \mathbf{B}^2 \mathbf{A} + \mathbf{I}_\mathbf{B} \mathbf{A}^2 + (\mathbf{I}_\mathbf{AB} - \mathbf{I}_\mathbf{A} \mathbf{I}_\mathbf{B}) \mathbf{A} - \mathbf{II}_\mathbf{A} \mathbf{B} + \mathbf{I}_\mathbf{A} \mathbf{AB} + \mathbf{I}_\mathbf{A} \mathbf{BA} + (\mathbf{I}_\mathbf{A}^2 \mathbf{B} - \mathbf{I}_\mathbf{A} \mathbf{I}_\mathbf{AB} + \mathbf{I}_\mathbf{B} \mathbf{II}_\mathbf{A}) \mathbf{I}_3$ [17, (4.22), p. 688], and applying it to $(\mathbf{A} + \mathbf{B} + \mathbf{C})^3$ results in the extended Cayley-Hamilton theorem $\mathbf{ABC} + \mathbf{CBA} + \mathbf{BCA} + \mathbf{ACB} + \mathbf{CAB} + \mathbf{BAC} = (\mathbf{I}_\mathbf{BC} - \mathbf{I}_\mathbf{B} \mathbf{I}_\mathbf{C}) \mathbf{A} + (\mathbf{I}_\mathbf{CA} - \mathbf{I}_\mathbf{C} \mathbf{I}_\mathbf{A}) \mathbf{B} + (\mathbf{I}_\mathbf{AB} - \mathbf{I}_\mathbf{A} \mathbf{I}_\mathbf{B}) \mathbf{C} + \mathbf{I}_\mathbf{A} \mathbf{BC} + \mathbf{I}_\mathbf{A} \mathbf{CB} + \mathbf{I}_\mathbf{B} \mathbf{CA} + \mathbf{I}_\mathbf{B} \mathbf{AC} + \mathbf{I}_\mathbf{C} \mathbf{BA} + \mathbf{I}_\mathbf{C} \mathbf{AB} + (\mathbf{I}_\mathbf{ABC} + \mathbf{I}_\mathbf{CBA} - \mathbf{I}_\mathbf{A} \mathbf{I}_\mathbf{BC} - \mathbf{I}_\mathbf{B} \mathbf{I}_\mathbf{CA} + \mathbf{I}_\mathbf{C} \mathbf{I}_\mathbf{AB} + \mathbf{I}_\mathbf{A} \mathbf{I}_\mathbf{B} \mathbf{I}_\mathbf{C}) \mathbf{I}_3$ [17, (4.21), p. 688]. The operator $\mathbf{I}_{(\cdot)} := \text{tr}(\cdot)$ denotes the trace (first invariant) of a tensor, and $\mathbf{I}_3 \in \mathbb{E}^{3 \times 3}$ is the order-2 identity tensor in the Euclidean space \mathbb{E}^3 . With respect to \mathbf{b} (2c), $\bar{\mathbf{S}}$ (2a) and $\bar{\mathbf{\Omega}}$ (2b), the Cayley-Hamilton theorem and its extensions [17] give [4, 6]

$$\mathbf{b}^3 = -\mathbf{II}_\mathbf{b} \mathbf{b} + \mathbf{III}_\mathbf{b} \mathbf{I}_3 \quad (3a)$$

$$\mathbf{b} \bar{\mathbf{S}} \mathbf{b} + \mathbf{b}^2 \bar{\mathbf{S}} + \bar{\mathbf{S}} \mathbf{b}^2 = -\mathbf{II}_\mathbf{b} \bar{\mathbf{S}} + \mathbf{I}_\mathbf{b} \bar{\mathbf{S}} \mathbf{b} + \mathbf{I}_\mathbf{b}^2 \bar{\mathbf{S}} \mathbf{I}_3 \quad (3b)$$

$$\mathbf{b} \bar{\mathbf{S}} \mathbf{b}^2 + \mathbf{b}^2 \bar{\mathbf{S}} \mathbf{b} = \mathbf{I}_\mathbf{b} \bar{\mathbf{S}} \mathbf{b}^2 + \mathbf{I}_\mathbf{b}^2 \bar{\mathbf{S}} \mathbf{b} - \mathbf{III}_\mathbf{b} \bar{\mathbf{S}} \quad (3c)$$

$$\mathbf{b} \bar{\mathbf{\Omega}} \mathbf{b} = -\mathbf{b}^2 \bar{\mathbf{\Omega}} - \bar{\mathbf{\Omega}} \mathbf{b}^2 - \mathbf{II}_\mathbf{b} \bar{\mathbf{\Omega}} \quad (3d)$$

$$\mathbf{b} \bar{\mathbf{\Omega}} \mathbf{b}^2 + \mathbf{b}^2 \bar{\mathbf{\Omega}} \mathbf{b} = -\mathbf{III}_\mathbf{b} \bar{\mathbf{\Omega}} \quad (3e)$$

The most general model for $\phi_{ij}^{(s; \mathfrak{V})}$ (1) is [3] a linear combination of \mathbf{b} (2c) and $\mathbf{b}^2 + \frac{2}{3} \mathbf{II}_\mathbf{b} \mathbf{I}_3$ ($\mathbf{I}_\mathbf{b}^2 = -2\mathbf{II}_\mathbf{b}$) with coefficients which are functions of the invariants $\mathbf{II}_\mathbf{b}$ and $\mathbf{III}_\mathbf{b}$, since by construction $\mathbf{I}_\mathbf{b} = 0$ (2c). The most general model for $\phi_{ij}^{(r; \mathfrak{V})}$ (1) is [4, 6, 7] a linear combination of the symmetric deviatoric tensors constructed from products of b_{ij} (2c), \bar{S}_{ij} (2a) and $\bar{\Omega}_{ij}$ (2b), which form a representation basis [18, 19] of deviatoric-symmetric-tensor-valued isotropic functions of these 3 tensors, omitting, because of the linearity requirement [3], elements nonlinear in the mean-velocity gradients (*ie* terms containing $\bar{\mathbf{S}}^{n_1} \bar{\mathbf{\Omega}}^{n_2}$ with $n_1 + n_2 > 1$). There are 2 different approaches to constructing a basis: polynomial bases [18] and functional bases [19]. Polynomial bases [18] are formed by all products of integer powers of the generating tensors which cannot be represented as a linear combination of the basis-elements using CH-reduction, *ie* identities obtained from the aforementioned Cayley-Hamilton theorem and its extensions [17], so that the representation coefficients are explicitly known polynomial (hence continuous) functions of the invariants. Functional bases [19] are potentially more compact, because they further reduce the elements of the corresponding polynomial basis by RE-reduction [20, pp. 380–382], *ie* by solving appropriate linear systems [19]. As a consequence, it can only be asserted [19] that the coefficients for representing a given product between integer powers of

the tensors generating the basis as a combination of the basis-elements are functions (piecewise rational) of the invariants, not necessarily continuous [19], which are not always explicitly known. For this reason [6] a polynomial representation basis is preferable, and in the present case this is [18, 6]

$$\mathfrak{B}[\mathbf{b}, \dot{\mathbf{S}}, \dot{\mathbf{\Omega}}] := \{\dot{\mathbf{T}}_{(n)}, n \in \{1, \dots, 8\}\} \text{ with}$$

$$\dot{\mathbf{T}}_{(1)} := \mathbf{b} = b_{ij} \vec{e}_i \otimes \vec{e}_j \quad (4a)$$

$$\dot{\mathbf{T}}_{(2)} := \mathbf{b}^2 - \frac{1}{3} \mathbf{I}_{\mathbf{b}} \mathbf{I}_3 = (b_{il} b_{lj} + \frac{2}{3} \Pi_{\mathbf{b}} \delta_{ij}) \vec{e}_i \otimes \vec{e}_j \quad (4b)$$

$$\dot{\mathbf{T}}_{(3)} := \dot{\mathbf{S}} := (\kappa \varepsilon^{-1}) \bar{S}_{ij} \vec{e}_i \otimes \vec{e}_j \quad (4c)$$

$$\dot{\mathbf{T}}_{(4)} := \mathbf{b} \dot{\mathbf{S}} + \dot{\mathbf{S}} \mathbf{b} - \frac{2}{3} \mathbf{I}_{\mathbf{b} \dot{\mathbf{S}}} \mathbf{I}_3 \stackrel{(2,4,5)}{=} -\frac{1}{2} \dot{\mathbf{P}}_{\dot{\mathbf{S}}} - \frac{2}{3} \dot{\mathbf{S}} \quad (4d)$$

$$\dot{\mathbf{T}}_{(5)} := \mathbf{b} \dot{\mathbf{\Omega}} - \dot{\mathbf{\Omega}} \mathbf{b} := (\kappa \varepsilon^{-1}) (\mathbf{b} \bar{\mathbf{\Omega}} - \bar{\mathbf{\Omega}} \mathbf{b}) \stackrel{(2,5)}{=} \frac{1}{2} \dot{\mathbf{P}}_{\dot{\mathbf{\Omega}}} \quad (4e)$$

$$\dot{\mathbf{T}}_{(6)} := \mathbf{b}^2 \dot{\mathbf{S}} + \dot{\mathbf{S}} \mathbf{b}^2 - \frac{2}{3} \mathbf{I}_{\mathbf{b}^2 \dot{\mathbf{S}}} \mathbf{I}_3 \quad (4f)$$

$$\dot{\mathbf{T}}_{(7)} := \mathbf{b}^2 \dot{\mathbf{\Omega}} - \dot{\mathbf{\Omega}} \mathbf{b}^2 \quad (4g)$$

$$\dot{\mathbf{T}}_{(8)} := \mathbf{b}^2 \dot{\mathbf{\Omega}} \mathbf{b} - \mathbf{b} \dot{\mathbf{\Omega}} \mathbf{b}^2 \quad (4h)$$

where the 8 tensors $\dot{\mathbf{T}}_{(n)}$ in (4) were made nondimensional by scaling with appropriate powers of the turbulence kinetic energy $k := \frac{1}{2} \overline{u'_\ell u'_\ell}$ and its dissipation-rate $\varepsilon := 2\nu \overline{\partial_{x_j} u'_i \partial_{x_j} u'_i}$.

The above basis, $\mathfrak{B}[\mathbf{b}, \dot{\mathbf{S}}, \dot{\mathbf{\Omega}}] := \{\dot{\mathbf{T}}_{(n)}, n \in \{1, \dots, 8\}\}$ (4), is the classical [6] representation basis in homogeneous turbulence, where $\dot{\mathbf{S}}$ and $\dot{\mathbf{\Omega}}$ are usually fixed inputs of the problem [2, 3].

Notice [6] that $\dot{\mathbf{T}}_{(8)}$ (4h) can in principle be projected by RE-reduction to obtain an irreducible functional representation basis [4], $\mathfrak{B}[\mathbf{b}, \dot{\mathbf{S}}, \dot{\mathbf{\Omega}}] \setminus \{\dot{\mathbf{T}}_{(8)}\}$, but the coefficients for this projection are not known explicitly. Furthermore, even if the projection were to be sought, on a value-by-value basis [20, pp. 380–382], the resulting representation coefficients would not necessarily be continuous functions of the invariants [19], and this makes functional representation bases awkward to use. The problem is even more acute in representation bases for wall turbulence [21], where the basic Gibson-Launder [22] rapid wall-echo model cannot be explicitly represented in the functional basis built by $[\mathbf{b}, \dot{\mathbf{S}}, \dot{\mathbf{\Omega}}]$ and the unit-vector in the normal-to-the-wall direction \vec{e}_n . Therefore, as noted in [6], polynomial representation bases are the best choice, because the increased number of basis-elements is justified by the possibility of explicit continuous representation of models, using the Cayley-Hamilton theorem and its extensions [17].

3 Production by strain or rotation

The tensor $\dot{\mathbf{T}}_{(4)}$ is related to strain-production (4d) and the tensor $\dot{\mathbf{T}}_{(5)}$ to rotation-production (4e). We can therefore construct an equivalent representation basis

$n \setminus m$	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0
4	0	0	$-\frac{4}{3}$	-2	0	0	0	0
5	0	0	0	0	2	0	0	0
6	$-\frac{4}{3} \mathbf{I}_{\mathbf{b} \dot{\mathbf{S}}}$	0	$4 \Pi_{\mathbf{b}}$	$-\frac{4}{3}$	0	2	0	0
7	0	0	0	0	0	0	2	0
8	0	0	0	0	$-2 \Pi_{\mathbf{b}}$	0	0	-2

Table 1. Matrix of coefficients a_{HTnm} for the representation $\dot{\mathbf{H}}_{(n)} = \sum_{m=1}^8 a_{\text{HTnm}} \dot{\mathbf{T}}_{(m)}$ (7) of the elements $\dot{\mathbf{H}}_{(n)} \in \mathfrak{B}_{(P)}[\mathbf{b}, \dot{\mathbf{S}}, \dot{\mathbf{P}}_{\dot{\mathbf{\Omega}}}; \dot{\mathbf{P}}_{\dot{\mathbf{S}}}]$ (5) as a linear combination of the elements $\dot{\mathbf{T}}_{(m)} \in \mathfrak{B}[\mathbf{b}, \dot{\mathbf{S}}, \dot{\mathbf{\Omega}}]$ (4).

$n \setminus m$	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0
4	0	0	$-\frac{2}{3}$	$-\frac{1}{2}$	0	0	0	0
5	0	0	0	0	$\frac{1}{2}$	0	0	0
6	$-\frac{1}{3} \frac{P_k}{\rho \varepsilon}$	0	$-2 \Pi_{\mathbf{b}} - \frac{4}{9}$	$-\frac{1}{3}$	0	$\frac{1}{2}$	0	0
7	0	0	0	0	0	0	$\frac{1}{2}$	0
8	0	0	0	0	$-\frac{1}{2} \Pi_{\mathbf{b}}$	0	0	$-\frac{1}{2}$

Table 2. Matrix of coefficients a_{THnm} for the representation $\dot{\mathbf{T}}_{(n)} = \sum_{m=1}^8 a_{\text{THnm}} \dot{\mathbf{H}}_{(m)}$ (7) of the elements $\dot{\mathbf{T}}_{(n)} \in \mathfrak{B}[\mathbf{b}, \dot{\mathbf{S}}, \dot{\mathbf{\Omega}}]$ (4) as a linear combination of the elements $\dot{\mathbf{H}}_{(m)} \in \mathfrak{B}_{(P)}[\mathbf{b}, \dot{\mathbf{S}}, \dot{\mathbf{P}}_{\dot{\mathbf{\Omega}}}; \dot{\mathbf{P}}_{\dot{\mathbf{S}}}]$ (5).

$$\mathfrak{B}_{(P)}[\mathbf{b}, \dot{\mathbf{S}}, \dot{\mathbf{P}}_{\dot{\mathbf{\Omega}}}; \dot{\mathbf{P}}_{\dot{\mathbf{S}}}] := \{\dot{\mathbf{H}}_{(n)}, n \in \{1, \dots, 8\}\} \text{ with}$$

$$\dot{\mathbf{H}}_{(1)} := \mathbf{b} \stackrel{(4a)}{=} \dot{\mathbf{T}}_{(1)} \quad (5a)$$

$$\dot{\mathbf{H}}_{(2)} := \mathbf{b}^2 - \frac{1}{3} \mathbf{I}_{\mathbf{b}} \mathbf{I}_3 \stackrel{(4b)}{=} \dot{\mathbf{T}}_{(2)} \quad (5b)$$

$$\dot{\mathbf{H}}_{(3)} := \dot{\mathbf{S}} \stackrel{(4c)}{=} \dot{\mathbf{T}}_{(3)} \quad (5c)$$

$$\dot{\mathbf{H}}_{(4)} := \dot{\mathbf{P}}_{\dot{\mathbf{S}}} := (\rho \varepsilon)^{-1} (P_{\dot{S}_{ij}} - \frac{2}{3} P_k \delta_{ij}) \vec{e}_i \otimes \vec{e}_j \stackrel{(4c, 4d)}{=} -2 \dot{\mathbf{T}}_{(4)} - \frac{4}{3} \dot{\mathbf{T}}_{(3)} \quad (5d)$$

$$\dot{\mathbf{H}}_{(5)} := \dot{\mathbf{P}}_{\dot{\mathbf{\Omega}}} := (\rho \varepsilon)^{-1} P_{\dot{\Omega}_{ij}} \vec{e}_i \otimes \vec{e}_j \stackrel{(4e)}{=} 2 \dot{\mathbf{T}}_{(5)} \quad (5e)$$

$$\dot{\mathbf{H}}_{(6)} := \mathbf{b} \dot{\mathbf{P}}_{\dot{\mathbf{S}}} + \dot{\mathbf{P}}_{\dot{\mathbf{S}}} \mathbf{b} - \frac{2}{3} \mathbf{I}_{\mathbf{b} \dot{\mathbf{P}}_{\dot{\mathbf{S}}}} \mathbf{I}_3 \stackrel{(2,4,3b)}{=} -\frac{4}{3} \mathbf{I}_{\mathbf{b} \dot{\mathbf{S}}} \dot{\mathbf{T}}_{(1)} + 4 \Pi_{\mathbf{b}} \dot{\mathbf{T}}_{(3)} - \frac{4}{3} \dot{\mathbf{T}}_{(4)} + 2 \dot{\mathbf{T}}_{(6)} \quad (5f)$$

$$\dot{\mathbf{H}}_{(7)} := \mathbf{b} \dot{\mathbf{P}}_{\dot{\mathbf{\Omega}}} + \dot{\mathbf{P}}_{\dot{\mathbf{\Omega}}} \mathbf{b} \stackrel{(2,4g,3d)}{=} 2 \dot{\mathbf{T}}_{(7)} \quad (5g)$$

$$\dot{\mathbf{H}}_{(8)} := \mathbf{b}^2 \dot{\mathbf{P}}_{\dot{\mathbf{\Omega}}} + \dot{\mathbf{P}}_{\dot{\mathbf{\Omega}}} \mathbf{b}^2 \stackrel{(2,4,3a)}{=} -2 \Pi_{\mathbf{b}} \dot{\mathbf{T}}_{(5)} - 2 \dot{\mathbf{T}}_{(8)} \quad (5h)$$

where the identities in (5) are obtained by direct computation from definitions (2c–2e), using the following relations between invariants, also obtained by direct computation from definitions (2)

$$\mathbf{I}_{\dot{\mathbf{P}}_{\dot{\Omega}}} \stackrel{(2)}{=} \mathbf{I}_{\dot{\mathbf{b}}\dot{\mathbf{P}}_{\dot{\Omega}}} \stackrel{(2)}{=} \mathbf{I}_{\dot{\mathbf{b}}^2\dot{\mathbf{P}}_{\dot{\Omega}}} \stackrel{(2)}{=} 0 \quad (6a)$$

$$2P_k \stackrel{(2f)}{=} \mathbf{I}_{\dot{\mathbf{P}}} \stackrel{(2d,2f)}{=} \mathbf{I}_{\dot{\mathbf{P}}_{\dot{\mathcal{S}}}} \stackrel{(2g,6a)}{=} \mathbf{I}_{\mathbf{D}} \quad (6b)$$

$$\mathbf{I}_{\dot{\mathbf{b}}\dot{\mathbf{S}}} \stackrel{(4d,6b)}{=} -\frac{1}{2} \frac{P_k}{\rho\epsilon} \quad (6c)$$

$$\mathbf{I}_{\dot{\mathbf{b}}\dot{\mathbf{D}}} \stackrel{(8c,8d,2)}{=} \mathbf{I}_{\dot{\mathbf{b}}\dot{\mathbf{P}}} \stackrel{(5d,8c,2)}{=} \mathbf{I}_{\dot{\mathbf{b}}\dot{\mathbf{P}}_{\dot{\mathcal{S}}}} \stackrel{(4c,5d,2)}{=} -4\mathbf{I}_{\dot{\mathbf{b}}^2\dot{\mathbf{S}}} - \frac{4}{3}\mathbf{I}_{\dot{\mathbf{b}}\dot{\mathbf{S}}} \quad (6d)$$

$$\mathbf{I}_{\dot{\mathbf{b}}^2\dot{\mathbf{S}}} \stackrel{(6d,6c)}{=} \frac{1}{6} \frac{P_k}{\rho\epsilon} - \frac{1}{4}\mathbf{I}_{\dot{\mathbf{b}}\dot{\mathbf{P}}_{\dot{\mathcal{S}}}} \quad (6e)$$

$$\mathbf{I}_{\dot{\mathbf{b}}^2\dot{\mathbf{P}}} \stackrel{(2,8,6)}{=} \mathbf{I}_{\dot{\mathbf{b}}^2\dot{\mathbf{D}}} \stackrel{(2,5,8)}{=} \mathbf{I}_{\dot{\mathbf{b}}^2\dot{\mathbf{P}}_{\dot{\mathcal{S}}}} \stackrel{(2,5,6)}{=} -\left(\frac{2}{9} + \frac{2}{3}\Pi_{\mathbf{b}}\right) \frac{P_k}{\rho\epsilon} + \frac{1}{3}\mathbf{I}_{\dot{\mathbf{b}}\dot{\mathbf{P}}_{\dot{\mathcal{S}}}} \quad (6f)$$

From the relations (6) we can readily identify the matrix of coefficients $\underline{\underline{a}}_{(\text{HT})} := [a_{(\text{HT})nm}] \in \mathbb{R}^{8 \times 8}$ (Tab. 1), and calculate its inverse $\underline{\underline{a}}_{(\text{TH})} := \underline{\underline{a}}_{(\text{HT})}^{-1}$ (Tab. 2), which relate the column-vectors of basis-elements $\underline{\underline{\mathbf{T}}} := [\dot{\mathbf{T}}_{(1)}, \dots, \dot{\mathbf{T}}_{(8)}]^T$ (4) and $\underline{\underline{\mathbf{H}}} := [\dot{\mathbf{H}}_{(1)}, \dots, \dot{\mathbf{H}}_{(8)}]^T$ (5)

$$\underline{\underline{\mathbf{H}}} = \underline{\underline{a}}_{(\text{HT})} \underline{\underline{\mathbf{T}}} \iff \underline{\underline{\mathbf{T}}} = \underbrace{\underline{\underline{a}}_{(\text{HT})}^{-1}}_{\underline{\underline{a}}_{(\text{TH})}} \underline{\underline{\mathbf{H}}} \quad (7)$$

The passage-matrix $\underline{\underline{a}}_{(\text{HT})}$ (Tab. 1), and its inverse $\underline{\underline{a}}_{(\text{TH})}$ (Tab. 2), can be used to systematically project models from one basis to another (§5).

4 Symmetric representation bases

Many early models [2, 8, 9] used the rate-of-production tensor P_{ij} (2f) and the tensor D_{ij} which we identified in (2g) as the difference between strain-production minus rotation-production. These 2 tensors are used by several researchers [13, 14], who sometimes [23] express models developed in the classical basis $\mathfrak{B}[\mathbf{b}, \dot{\mathbf{S}}, \dot{\Omega}]$ (4) in terms of P_{ij} (2f) and D_{ij} (2g). If we consider the tensors P_{ij} (2f) and D_{ij} (2g) representative of the influence of mean-velocity gradients $\partial_{x_j} \bar{u}_i$ on the ϕ_{ij} (1), since it is P_{ij} (2f), and not \tilde{S}_{ij} (2a) alone, which appears directly in the transport equations for the Reynolds-stresses [4], we may construct a representation basis generated by b_{ij} (2c) and these 2 tensors, $\mathfrak{B}[\mathbf{b}, \dot{\mathbf{P}}, \dot{\mathbf{D}}] := \{\dot{\mathbf{J}}_{(n)}, n \in \{1, \dots, 8\}\}$ with

$$\dot{\mathbf{J}}_{(1)} := \mathbf{b} \stackrel{(2a)}{=} \dot{\mathbf{T}}_{(1)} \quad (8a)$$

$$\dot{\mathbf{J}}_{(2)} := \mathbf{b}^2 - \frac{1}{3}\mathbf{I}_{\mathbf{b}} \mathbf{I}_3 \stackrel{(2b)}{=} \dot{\mathbf{T}}_{(2)} \quad (8b)$$

$$\begin{aligned} \dot{\mathbf{J}}_{(3)} := \dot{\mathbf{P}} &:= (\rho\epsilon)^{-1} (P_{ij} - \frac{2}{3}P_k\delta_{ij}) \vec{e}_i \otimes \vec{e}_j \stackrel{(2f,5d,5e)}{=} \\ &- \frac{4}{3}\dot{\mathbf{T}}_{(3)} - 2\dot{\mathbf{T}}_{(4)} + 2\dot{\mathbf{T}}_{(5)} \end{aligned} \quad (8c)$$

$$\begin{aligned} \dot{\mathbf{J}}_{(4)} := \dot{\mathbf{D}} &:= (\rho\epsilon)^{-1} (D_{ij} - \frac{2}{3}P_k\delta_{ij}) \vec{e}_i \otimes \vec{e}_j \stackrel{(2g,5d,5e)}{=} \\ &- \frac{4}{3}\dot{\mathbf{T}}_{(3)} - 2\dot{\mathbf{T}}_{(4)} - 2\dot{\mathbf{T}}_{(5)} \end{aligned} \quad (8d)$$

$$\begin{aligned} \dot{\mathbf{J}}_{(5)} := \mathbf{b}\dot{\mathbf{P}} + \dot{\mathbf{P}}\mathbf{b} - \frac{2}{3}\mathbf{I}_{\mathbf{b}}\dot{\mathbf{P}}_{\dot{\mathcal{S}}} \mathbf{I}_3 &\stackrel{(2f,5f,5g)}{=} \\ &- \frac{4}{3}\mathbf{I}_{\dot{\mathbf{b}}\dot{\mathbf{S}}} \dot{\mathbf{T}}_{(1)} + 4\Pi_{\mathbf{b}} \dot{\mathbf{T}}_{(3)} - \frac{4}{3}\dot{\mathbf{T}}_{(4)} + 2\dot{\mathbf{T}}_{(6)} + 2\dot{\mathbf{T}}_{(7)} \end{aligned} \quad (8e)$$

$$\begin{aligned} \dot{\mathbf{J}}_{(6)} := \mathbf{b}\dot{\mathbf{D}} + \dot{\mathbf{D}}\mathbf{b} - \frac{2}{3}\mathbf{I}_{\mathbf{b}}\dot{\mathbf{D}}_{\dot{\mathcal{S}}} \mathbf{I}_3 &\stackrel{(2g,5f,5g)}{=} \\ &- \frac{4}{3}\mathbf{I}_{\dot{\mathbf{b}}\dot{\mathbf{S}}} \dot{\mathbf{T}}_{(1)} + 4\Pi_{\mathbf{b}} \dot{\mathbf{T}}_{(3)} - \frac{4}{3}\dot{\mathbf{T}}_{(4)} + 2\dot{\mathbf{T}}_{(6)} - 2\dot{\mathbf{T}}_{(7)} \end{aligned} \quad (8f)$$

$$\begin{aligned} \dot{\mathbf{J}}_{(7)} := \mathbf{b}^2\dot{\mathbf{P}} + \dot{\mathbf{P}}\mathbf{b}^2 - \frac{2}{3}\mathbf{I}_{\mathbf{b}^2\dot{\mathbf{P}}} \mathbf{I}_3 &\stackrel{(2f,5h,9g)}{=} \\ &- 2\mathbf{I}_{\dot{\mathbf{b}}^2\dot{\mathbf{S}}} \dot{\mathbf{T}}_{(1)} + \frac{2}{3}\mathbf{I}_{\dot{\mathbf{b}}\dot{\mathbf{S}}} \dot{\mathbf{T}}_{(2)} - 2\Pi_{\mathbf{b}} \dot{\mathbf{T}}_{(3)} \\ &+ 2\Pi_{\mathbf{b}} \dot{\mathbf{T}}_{(4)} - 2\Pi_{\mathbf{b}} \dot{\mathbf{T}}_{(5)} - \frac{4}{3}\dot{\mathbf{T}}_{(6)} - 2\dot{\mathbf{T}}_{(8)} \end{aligned} \quad (8g)$$

$$\begin{aligned} \dot{\mathbf{J}}_{(8)} := \mathbf{b}^2\dot{\mathbf{D}} + \dot{\mathbf{D}}\mathbf{b}^2 - \frac{2}{3}\mathbf{I}_{\mathbf{b}^2\dot{\mathbf{P}}} \mathbf{I}_3 &\stackrel{(2g,5h,9g)}{=} \\ &- 2\mathbf{I}_{\dot{\mathbf{b}}^2\dot{\mathbf{S}}} \dot{\mathbf{T}}_{(1)} + \frac{2}{3}\mathbf{I}_{\dot{\mathbf{b}}\dot{\mathbf{S}}} \dot{\mathbf{T}}_{(2)} - 2\Pi_{\mathbf{b}} \dot{\mathbf{T}}_{(3)} \\ &+ 2\Pi_{\mathbf{b}} \dot{\mathbf{T}}_{(4)} + 2\Pi_{\mathbf{b}} \dot{\mathbf{T}}_{(5)} - \frac{4}{3}\dot{\mathbf{T}}_{(6)} + 2\dot{\mathbf{T}}_{(8)} \end{aligned} \quad (8h)$$

where the relations with the elements of $\mathfrak{B}[\mathbf{b}, \dot{\mathbf{S}}, \dot{\Omega}] := \{\dot{\mathbf{T}}_{(n)}, n \in \{1, \dots, 8\}\}$ (4) are obtained by direct computation using the expressions (6) for the invariants, and the aforementioned Cayley-Hamilton theorem and its extensions [17]. Alternatively we can use $P_{\tilde{S}ij} \stackrel{(2f,2g)}{=} \frac{1}{2}(P_{ij} + D_{ij})$ and $P_{\tilde{\Omega}ij} \stackrel{(2f,2g)}{=} \frac{1}{2}(P_{ij} - D_{ij})$ to obtain an equivalent representation basis generated by the 3 symmetric tensors \mathbf{b} , $\dot{\mathbf{P}}_{\dot{\mathcal{S}}}$ and $\dot{\mathbf{P}}_{\dot{\Omega}}$, $\mathfrak{B}[\mathbf{b}, \dot{\mathbf{P}}_{\dot{\mathcal{S}}}, \dot{\mathbf{P}}_{\dot{\Omega}}] := \{\dot{\mathbf{F}}_{(n)}, n \in \{1, \dots, 8\}\}$ with

$$\dot{\mathbf{F}}_{(1)} := \mathbf{b} \stackrel{(2a)}{=} \dot{\mathbf{T}}_{(1)} \quad (9a)$$

$$\dot{\mathbf{F}}_{(2)} := \mathbf{b}^2 - \frac{1}{3}\mathbf{I}_{\mathbf{b}} \mathbf{I}_3 \stackrel{(2b)}{=} \dot{\mathbf{T}}_{(2)} \quad (9b)$$

$$\dot{\mathbf{F}}_{(3)} := \dot{\mathbf{P}}_{\dot{\mathcal{S}}} \stackrel{(5d)}{=} \dot{\mathbf{H}}_{(4)} \quad (9c)$$

$$\dot{\mathbf{F}}_{(4)} := \dot{\mathbf{P}}_{\dot{\Omega}} \stackrel{(5e)}{=} \dot{\mathbf{H}}_{(5)} \quad (9d)$$

$$\dot{\mathbf{F}}_{(5)} := \mathbf{b}\dot{\mathbf{P}}_{\dot{\mathcal{S}}} + \dot{\mathbf{P}}_{\dot{\mathcal{S}}}\mathbf{b} - \frac{2}{3}\mathbf{I}_{\mathbf{b}}\dot{\mathbf{P}}_{\dot{\mathcal{S}}} \mathbf{I}_3 \stackrel{(5f)}{=} \dot{\mathbf{H}}_{(6)} \quad (9e)$$

$$\dot{\mathbf{F}}_{(6)} := \mathbf{b}\dot{\mathbf{P}}_{\dot{\Omega}} + \dot{\mathbf{P}}_{\dot{\Omega}}\mathbf{b} \stackrel{(5g)}{=} \dot{\mathbf{H}}_{(7)} \quad (9f)$$

$$\begin{aligned} \dot{\mathbf{F}}_{(7)} := \mathbf{b}^2\dot{\mathbf{P}}_{\dot{\mathcal{S}}} + \dot{\mathbf{P}}_{\dot{\mathcal{S}}}\mathbf{b}^2 - \frac{2}{3}\mathbf{I}_{\mathbf{b}^2\dot{\mathbf{P}}_{\dot{\mathcal{S}}}} \mathbf{I}_3 &\stackrel{(3a,3c,4)}{=} \\ &- 2\mathbf{I}_{\dot{\mathbf{b}}^2\dot{\mathbf{S}}} \dot{\mathbf{T}}_{(1)} + \frac{2}{3}\mathbf{I}_{\dot{\mathbf{b}}\dot{\mathbf{S}}} \dot{\mathbf{T}}_{(2)} \\ &- 2\Pi_{\mathbf{b}} \dot{\mathbf{T}}_{(3)} + 2\Pi_{\mathbf{b}} \dot{\mathbf{T}}_{(4)} - \frac{4}{3}\dot{\mathbf{T}}_{(6)} \end{aligned} \quad (9g)$$

$$\dot{\mathbf{F}}_{(8)} := \mathbf{b}^2\dot{\mathbf{P}}_{\dot{\Omega}} + \dot{\mathbf{P}}_{\dot{\Omega}}\mathbf{b}^2 \stackrel{(5h)}{=} \dot{\mathbf{H}}_{(8)} \quad (9h)$$

where (9g) is again obtained by direct computation, using

$n \backslash m$	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0
3	0	0	$-\frac{4}{3}$	-2	+2	0	0	0
4	0	0	$-\frac{4}{3}$	-2	-2	0	0	0
5	$-\frac{4}{3}\mathbf{I}_{\mathbf{b}\mathbf{\hat{S}}}$	0	$4\Pi_{\mathbf{b}}$	$-\frac{4}{3}$	0	+2	+2	0
6	$-\frac{4}{3}\mathbf{I}_{\mathbf{b}\mathbf{\hat{S}}}$	0	$4\Pi_{\mathbf{b}}$	$-\frac{4}{3}$	0	+2	-2	0
7	$-2\mathbf{I}_{\mathbf{b}\mathbf{\hat{S}}}$	$\frac{2}{3}\mathbf{I}_{\mathbf{b}\mathbf{\hat{S}}}$	$-2\Pi_{\mathbf{b}}$	$+2\Pi_{\mathbf{b}}$	$-2\Pi_{\mathbf{b}}$	$-\frac{4}{3}$	0	-2
8	$-2\mathbf{I}_{\mathbf{b}\mathbf{\hat{S}}}$	$\frac{2}{3}\mathbf{I}_{\mathbf{b}\mathbf{\hat{S}}}$	$-2\Pi_{\mathbf{b}}$	$+2\Pi_{\mathbf{b}}$	$+2\Pi_{\mathbf{b}}$	$-\frac{4}{3}$	0	+2

Table 3. Matrix of coefficients $a_{\mathbf{J}\mathbf{T}_{nm}}$ for the representation $\mathbf{J}_{(n)} = \sum_{m=1}^8 a_{\mathbf{J}\mathbf{T}_{nm}} \mathbf{T}_{(m)}^*$ (10a) of the elements $\mathbf{J}_{(n)} \in \mathfrak{B}[\mathbf{b}, \mathbf{\hat{P}}, \mathbf{\hat{D}}]$ (8) as a linear combination of the elements $\mathbf{T}_{(m)}^* \in \mathfrak{B}[\mathbf{b}, \mathbf{\hat{S}}, \mathbf{\hat{\Omega}}]$ (4).

$n \backslash m$	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0
3	0	0	$-\frac{4}{3}$	-2	0	0	0	0
4	0	0	0	0	2	0	0	0
5	$-\frac{4}{3}\mathbf{I}_{\mathbf{b}\mathbf{\hat{S}}}$	0	$4\Pi_{\mathbf{b}}$	$-\frac{4}{3}$	0	2	0	0
6	0	0	0	0	0	0	2	0
7	$-2\mathbf{I}_{\mathbf{b}\mathbf{\hat{S}}}$	$\frac{2}{3}\mathbf{I}_{\mathbf{b}\mathbf{\hat{S}}}$	$-2\Pi_{\mathbf{b}}$	$2\Pi_{\mathbf{b}}$	0	$-\frac{4}{3}$	0	0
8	0	0	0	0	$-2\Pi_{\mathbf{b}}$	0	0	-2

Table 4. Matrix of coefficients $a_{\mathbf{F}\mathbf{T}_{nm}}$ for the representation $\mathbf{F}_{(n)} = \sum_{m=1}^8 a_{\mathbf{F}\mathbf{T}_{nm}} \mathbf{T}_{(m)}^*$ (10b) of the elements $\mathbf{F}_{(n)} \in \mathfrak{B}[\mathbf{b}, \mathbf{\hat{P}}, \mathbf{\hat{\Omega}}]$ (9) as a linear combination of the elements $\mathbf{T}_{(m)}^* \in \mathfrak{B}[\mathbf{b}, \mathbf{\hat{S}}, \mathbf{\hat{\Omega}}]$ (4).

(2,4,5,6).

Each of the bases $\mathfrak{B}[\mathbf{b}, \mathbf{\hat{P}}, \mathbf{\hat{D}}]$ (8) or $\mathfrak{B}[\mathbf{b}, \mathbf{\hat{P}}, \mathbf{\hat{\Omega}}]$ (9) is generated by 3 symmetric deviatoric tensors. Therefore, under the constraint of linearity in mean-velocity gradients [3], $\mathfrak{B}[\mathbf{b}, \mathbf{\hat{P}}, \mathbf{\hat{D}}]$ (8) is symmetric in $\mathbf{\hat{P}}$ (8c,2f) and $\mathbf{\hat{D}}$ (8d,2g), and $\mathfrak{B}[\mathbf{b}, \mathbf{\hat{P}}, \mathbf{\hat{\Omega}}]$ (9) is symmetric in $\mathbf{\hat{P}}$ (5d,2d) and $\mathbf{\hat{\Omega}}$ (5e,2e). On the contrary, the classical representation basis $\mathfrak{B}[\mathbf{b}, \mathbf{\hat{S}}, \mathbf{\hat{\Omega}}]$ (4), being built by the 2 symmetric tensors \mathbf{b} (2c) and $\mathbf{\hat{S}}$ (2a), and the antisymmetric tensor $\mathbf{\hat{\Omega}}$ (2b), contains the element $\mathbf{T}_{(8)}^* := \mathbf{b}^2 \mathbf{\hat{\Omega}} \mathbf{b} - \mathbf{b} \mathbf{\hat{\Omega}} \mathbf{b}^2$ (4h), but not the nondimensional deviatoric projection of $\mathbf{b} \mathbf{\hat{S}} \mathbf{b}^2 + \mathbf{b}^2 \mathbf{\hat{S}} \mathbf{b} \stackrel{(3c)}{=} \mathbf{I}_{\mathbf{b}\mathbf{\hat{S}}} \mathbf{b}^2 + \mathbf{I}_{\mathbf{b}^2 \mathbf{\hat{S}}} \mathbf{b} - \Pi_{\mathbf{b}} \mathbf{\hat{S}}$, which is CH-reducible by (3c). As a consequence, the interpretation $\mathfrak{B}_{(P)}[\mathbf{b}, \mathbf{\hat{S}}, \mathbf{\hat{P}}, \mathbf{\hat{\Omega}}]$ (5) of the classical representation basis in terms of strain-production $P_{\mathbf{\hat{S}}_{ij}}$ (2d) and rotation-production $P_{\mathbf{\hat{\Omega}}_{ij}}$ (2e) is not symmetric in $\mathbf{\hat{P}}$ (5d,2d) and $\mathbf{\hat{\Omega}}$ (5e,2e).

The basis-elements of $\mathfrak{B}[\mathbf{b}, \mathbf{\hat{P}}, \mathbf{\hat{D}}]$ (8) or $\mathfrak{B}[\mathbf{b}, \mathbf{\hat{P}}, \mathbf{\hat{\Omega}}]$ (9)

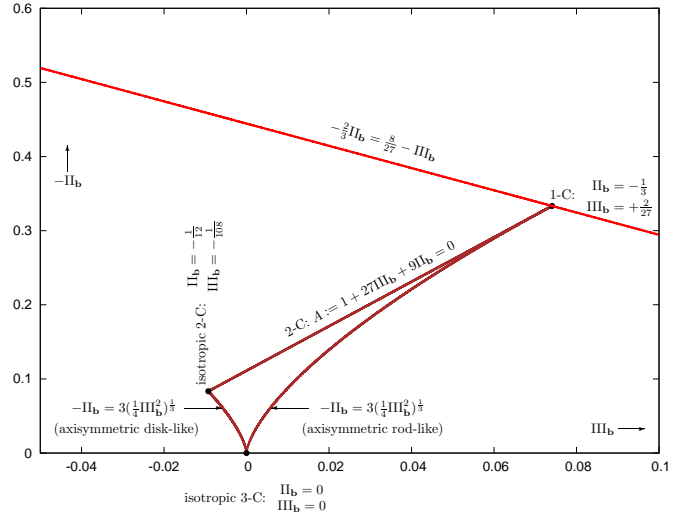


Fig. 1. Locus, in the $(\Pi_{\mathbf{b}}, -\Pi_{\mathbf{b}})$ -plane of the invariants of the anisotropy tensor [3], of the condition $-\frac{2}{3}\Pi_{\mathbf{b}} = \frac{8}{27} - \Pi_{\mathbf{b}}$ for which the representation $\mathbf{T}_{(n)}^* = \sum_{m=1}^8 a_{\mathbf{T}\mathbf{F}_{nm}} \mathbf{F}_{(m)}^*$ (10b) of the mean strain-rate tensor $\mathbf{\hat{S}}$ in $\mathfrak{B}[\mathbf{b}, \mathbf{\hat{P}}, \mathbf{\hat{\Omega}}]$ would be singular (the line $-\frac{2}{3}\Pi_{\mathbf{b}} = \frac{8}{27} - \Pi_{\mathbf{b}}$ lies outside of Lumley's realizability triangle [3, 24], but includes the 1-C point).

do not contain explicitly \bar{S}_{ij} (2a). Hence, the practical use of the symmetric bases $\mathfrak{B}[\mathbf{b}, \mathbf{\hat{P}}, \mathbf{\hat{D}}]$ (8) or $\mathfrak{B}[\mathbf{b}, \mathbf{\hat{P}}, \mathbf{\hat{\Omega}}]$ (9), hinges upon their ability to represent models containing explicitly \bar{S}_{ij} (2a) in their formulation, or, in terms of bases, on the possibility to project the classical representation basis $\mathfrak{B}[\mathbf{b}, \mathbf{\hat{S}}, \mathbf{\hat{\Omega}}]$ (4) onto these alternative bases. We may readily identify, from (4,5,6,8), the matrix of coefficients $\underline{a}_{(\mathbf{T})} := [a_{(\mathbf{T})_{nm}}] \in \mathbb{R}^{8 \times 8}$ (Tab. 3), representing the column-vector $\underline{\mathbf{J}} := [\mathbf{J}_{(1)}, \dots, \mathbf{J}_{(8)}]^T$ (8) in terms of the column-vector of the classical basis-elements $\underline{\mathbf{T}} := [\mathbf{T}_{(1)}, \dots, \mathbf{T}_{(8)}]^T$ (4), and from (4,5,6,9), the matrix of coefficients $\underline{a}_{(\mathbf{F})} := [a_{(\mathbf{F})_{nm}}] \in \mathbb{R}^{8 \times 8}$ (Tab. 4), representing the column-vector $\underline{\mathbf{F}} := [\mathbf{F}_{(1)}, \dots, \mathbf{F}_{(8)}]^T$ (5) in terms of the column-vector of the classical basis-elements $\underline{\mathbf{T}} := [\mathbf{T}_{(1)}, \dots, \mathbf{T}_{(8)}]^T$ (4),

$$\underline{\mathbf{J}} = \underline{a}_{\mathbf{T}} \underline{\mathbf{T}} \iff \underline{\mathbf{T}} = \underline{a}_{\mathbf{T}\mathbf{J}} \underline{\mathbf{J}} \quad (10a)$$

$$\underline{\mathbf{F}} = \underline{a}_{\mathbf{F}\mathbf{T}} \underline{\mathbf{T}} \iff \underline{\mathbf{T}} = \underline{a}_{\mathbf{T}\mathbf{F}} \underline{\mathbf{F}} \quad (10b)$$

The inverse matrices $\underline{a}_{(\mathbf{T}\mathbf{F})} := \underline{a}_{\mathbf{F}\mathbf{T}}^{-1}$ (Tab. 5) and $\underline{a}_{(\mathbf{T}\mathbf{J})} := \underline{a}_{\mathbf{T}\mathbf{J}}^{-1}$ (Tab. 6), which represent the column-vector of the elements of the classical basis in the 2 symmetric bases (10), are readily obtained by straightforward inversion using symbolic calculation [25]. The coefficients (Tabs. 5,6) for the representation of the elements of the classical basis $\mathfrak{B}[\mathbf{b}, \mathbf{\hat{S}}, \mathbf{\hat{\Omega}}]$ (4) in $\mathfrak{B}[\mathbf{b}, \mathbf{\hat{P}}, \mathbf{\hat{D}}]$ (8) and $\mathfrak{B}[\mathbf{b}, \mathbf{\hat{P}}, \mathbf{\hat{\Omega}}]$ (9) are rational (hence continuous except at poles) functions of the invariants. All of the

$n \setminus m$	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0
3	$-\frac{1}{18}\frac{P_k}{\rho\epsilon} - \frac{1}{4}\mathbf{I}_{\mathbf{b}\dot{\mathbf{P}}}$	$\frac{1}{6}\frac{P_k}{\rho\epsilon}$	$\frac{\frac{1}{2}\Pi_{\mathbf{b}} - \frac{2}{9}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	0	$\frac{\frac{1}{3}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	0	$\frac{\frac{1}{2}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	0
4	$\frac{\frac{1}{27}\frac{P_k}{\rho\epsilon} + \frac{1}{6}\mathbf{I}_{\mathbf{b}\dot{\mathbf{P}}}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$-\frac{1}{9}\frac{P_k}{\rho\epsilon}$	$\frac{-\frac{2}{3}\Pi_{\mathbf{b}} + \frac{1}{2}\text{III}_{\mathbf{b}}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	0	$\frac{-\frac{2}{9}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	0	$\frac{-\frac{1}{3}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	0
5	0	0	0	$\frac{1}{2}$	0	0	0	0
6	$\frac{(-\frac{1}{9}\Pi_{\mathbf{b}} + \frac{1}{3}\text{III}_{\mathbf{b}} - \frac{2}{27})\frac{P_k}{\rho\epsilon} + (\frac{1}{2}\Pi_{\mathbf{b}} + \frac{1}{9})\mathbf{I}_{\mathbf{b}\dot{\mathbf{P}}}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{(-\frac{1}{3}\Pi_{\mathbf{b}} - \frac{2}{27})\frac{P_k}{\rho\epsilon}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{-\Pi_{\mathbf{b}}^2 + \frac{1}{3}\text{III}_{\mathbf{b}}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	0	$\frac{-\frac{1}{3}\Pi_{\mathbf{b}} - \frac{1}{2}\text{III}_{\mathbf{b}}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	0	$\frac{-\Pi_{\mathbf{b}} - \frac{2}{9}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	0
7	0	0	0	0	0	$\frac{1}{2}$	0	0
8	0	0	0	$-\frac{1}{2}\Pi_{\mathbf{b}}$	0	0	0	$-\frac{1}{2}$

Table 5. Matrix of coefficients a_{TFnm} for the representation $\hat{\mathbf{T}}_{(n)} = \sum_{m=1}^8 a_{\text{TFnm}} \hat{\mathbf{F}}_{(m)}$ (10b) of the elements $\hat{\mathbf{T}}_{(n)} \in \mathfrak{B}[\mathbf{b}, \dot{\mathbf{S}}, \dot{\mathbf{\Omega}}]$ (4) as a linear combination of the elements $\hat{\mathbf{F}}_{(m)} \in \mathfrak{B}[\mathbf{b}, \dot{\mathbf{P}}, \dot{\mathbf{P}}_{\dot{\mathbf{\Omega}}}]$ (9).

$n \setminus m$	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0
3	$-\frac{1}{18}\frac{P_k}{\rho\epsilon} - \frac{1}{4}\mathbf{I}_{\mathbf{b}\dot{\mathbf{P}}}$	$\frac{1}{6}\frac{P_k}{\rho\epsilon}$	$\frac{\frac{1}{4}\Pi_{\mathbf{b}} - \frac{1}{9}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{\frac{1}{4}\Pi_{\mathbf{b}} - \frac{1}{9}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{\frac{1}{6}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{\frac{1}{6}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{\frac{1}{4}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{\frac{1}{4}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$
4	$\frac{\frac{1}{27}\frac{P_k}{\rho\epsilon} + \frac{1}{6}\mathbf{I}_{\mathbf{b}\dot{\mathbf{P}}}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$-\frac{1}{9}\frac{P_k}{\rho\epsilon}$	$\frac{-\frac{1}{4}\Pi_{\mathbf{b}} + \frac{1}{4}\text{III}_{\mathbf{b}}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{-\frac{1}{4}\Pi_{\mathbf{b}} + \frac{1}{4}\text{III}_{\mathbf{b}}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{-\frac{1}{9}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{-\frac{1}{9}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{-\frac{1}{6}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{-\frac{1}{6}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$
5	0	0	$+\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	0
6	$\frac{(-\frac{1}{9}\Pi_{\mathbf{b}} + \frac{1}{3}\text{III}_{\mathbf{b}} - \frac{2}{27})\frac{P_k}{\rho\epsilon} + (\frac{1}{2}\Pi_{\mathbf{b}} + \frac{1}{9})\mathbf{I}_{\mathbf{b}\dot{\mathbf{P}}}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{(-\frac{1}{3}\Pi_{\mathbf{b}} - \frac{2}{27})\frac{P_k}{\rho\epsilon}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{-\frac{1}{4}\Pi_{\mathbf{b}}^2 + \frac{1}{6}\text{III}_{\mathbf{b}}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{-\frac{1}{4}\Pi_{\mathbf{b}}^2 + \frac{1}{6}\text{III}_{\mathbf{b}}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{-\frac{1}{6}\Pi_{\mathbf{b}} - \frac{1}{4}\text{III}_{\mathbf{b}}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{-\frac{1}{6}\Pi_{\mathbf{b}} - \frac{1}{4}\text{III}_{\mathbf{b}}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{-\frac{1}{4}\Pi_{\mathbf{b}} - \frac{1}{9}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$	$\frac{-\frac{1}{4}\Pi_{\mathbf{b}} - \frac{1}{9}}{\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}}$
7	0	0	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	0	0
8	0	0	$-\frac{1}{4}\Pi_{\mathbf{b}}$	$+\frac{1}{4}\Pi_{\mathbf{b}}$	0	0	$-\frac{1}{4}$	$\frac{1}{4}$

Table 6. Matrix of coefficients a_{TJnm} for the representation $\hat{\mathbf{T}}_{(n)} = \sum_{m=1}^8 a_{\text{TJnm}} \hat{\mathbf{J}}_{(m)}$ (10a) of the elements $\hat{\mathbf{T}}_{(n)} \in \mathfrak{B}[\mathbf{b}, \dot{\mathbf{S}}, \dot{\mathbf{\Omega}}]$ (4) as a linear combination of the elements $\hat{\mathbf{J}}_{(m)} \in \mathfrak{B}[\mathbf{b}, \dot{\mathbf{P}}, \dot{\mathbf{D}}]$ (8).

coefficients of $\underline{a}_{(\text{TF})}$ (Tab. 5) and $\underline{a}_{(\text{TJ})}$ (Tab. 6) which could become singular have the same denominator $\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}$ (Tabs. 5,6). It turns out that $\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}} \neq 0$, because the line $-\Pi_{\mathbf{b}} = \frac{4}{9} - \frac{3}{2}\text{III}_{\mathbf{b}}$ lies outside of the realizability triangle [3] of the invariants of \mathbf{b} , $\Pi_{\mathbf{b}}$ and $\text{III}_{\mathbf{b}}$ (Fig. 1), with the unique exception of the 1-C point ($\Pi_{\mathbf{b}} = -\frac{1}{3}$, $\text{III}_{\mathbf{b}} = \frac{2}{27}$). Hence, the projection (10) is valid for any realizable [3] anisotropy tensor b_{ij} , except at the 1-C point. Using the representation coefficients $a_{(\text{TF})3m}$ (Tab. 5) and $a_{(\text{TJ})3m}$ (Tab. 6), the strain-rate tensor is explicitly represented in these bases

$$\left(\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}\right) \frac{k}{\epsilon} \bar{S}_{ij} = \left(-\frac{1}{18}\frac{P_k}{\rho\epsilon} - \frac{1}{4}\mathbf{I}_{\mathbf{b}\dot{\mathbf{P}}}\right) b_{ij}$$

$$\begin{aligned} & + \frac{1}{6}\frac{P_k}{\rho\epsilon} (b_{ik}b_{\ell j} + \frac{2}{3}\Pi_{\mathbf{b}}\delta_{ij}) \\ & + \left(\frac{1}{2}\Pi_{\mathbf{b}} - \frac{2}{9}\right) \frac{1}{\rho\epsilon} (P_{\bar{S}_{ij}} - \frac{2}{3}P_k\delta_{ij}) \\ & + \frac{1}{3}\dot{F}_{(5)ij} + \frac{1}{2}\dot{F}_{(7)ij} \end{aligned} \quad (11a)$$

$$\begin{aligned} \left(\frac{8}{27} + \frac{2}{3}\Pi_{\mathbf{b}} - \text{III}_{\mathbf{b}}\right) \frac{k}{\epsilon} \bar{S}_{ij} = & \left(-\frac{1}{18}\frac{P_k}{\rho\epsilon} - \frac{1}{4}\mathbf{I}_{\mathbf{b}\dot{\mathbf{P}}}\right) b_{ij} \\ & + \frac{1}{6}\frac{P_k}{\rho\epsilon} (b_{ik}b_{\ell j} + \frac{2}{3}\Pi_{\mathbf{b}}\delta_{ij}) \\ & + \left(\frac{1}{4}\Pi_{\mathbf{b}} - \frac{1}{9}\right) \frac{1}{\rho\epsilon} (P_{ij} - \frac{2}{3}P_k\delta_{ij}) \\ & + \left(\frac{1}{4}\Pi_{\mathbf{b}} - \frac{1}{9}\right) \frac{1}{\rho\epsilon} (D_{ij} - \frac{2}{3}P_k\delta_{ij}) \\ & + \frac{1}{6}\dot{J}_{(5)ij} + \frac{1}{6}\dot{J}_{(6)ij} \\ & + \frac{1}{4}\dot{J}_{(7)ij} + \frac{1}{4}\dot{J}_{(8)ij} \end{aligned} \quad (11b)$$

The representation (11a) of \bar{S}_{ij} in $\mathfrak{B}[\mathbf{b}, \bar{\mathbf{P}}_{\bar{S}}, \bar{\mathbf{P}}_{\bar{\Omega}}]$ (9) is more compact than the representation (11b) in $\mathfrak{B}[\mathbf{b}, \bar{\mathbf{P}}, \bar{\mathbf{D}}]$ (8), because in the first case the strain-related terms ($\bar{F}_{(3)ij}$, $\bar{F}_{(5)ij}$, $\bar{F}_{(7)ij}$; (9)) are separated from the rotation-related terms ($\bar{F}_{(4)ij}$, $\bar{F}_{(6)ij}$, $\bar{F}_{(8)ij}$; (9)), whereas in the second case they are coupled because of the identities (2f, 2g).

5 Change-of-basis relations

Since different model proposals in the literature use different bases [4, 6, 7, 8, 9, 11, 12, 13, 14] it is necessary to establish projection rules for the representation coefficients of various models in different bases. Any expression (closure) for ϕ_{ij} can be represented equivalently in any of the bases, with linear relations between representation coefficients. Consider 2 bases with column-vectors of basis-elements $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$, related by

$$\bar{\mathbf{A}} = \underline{\underline{a}}_{(AB)} \bar{\mathbf{B}} \iff \bar{\mathbf{B}} = \underline{\underline{a}}_{(AB)}^{-1} \bar{\mathbf{A}} \quad (12a)$$

where the existence of the inverse matrix $\underline{\underline{a}}_{(BA)} := \underline{\underline{a}}_{(AB)}^{-1} \in \mathbb{R}^{8 \times 8}$ is equivalent to the linear independence of the basis-elements, and hence to the fact that both $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ form representation bases. Denoting $\underline{\underline{c}}_{\phi A} \in \mathbb{R}^8$ and $\underline{\underline{c}}_{\phi B} \in \mathbb{R}^8$ the representation coefficients of ϕ in each basis, we readily have

$$\begin{aligned} (\rho \varepsilon)^{-1} \phi &= \sum_{n=1}^8 c_{\phi A_n} \bar{\mathbf{A}}_{(n)} = \underline{\underline{c}}_{\phi A}^T \bar{\mathbf{A}} \\ &\stackrel{(12a)}{=} \underline{\underline{c}}_{\phi A}^T \underline{\underline{a}}_{(AB)} \bar{\mathbf{B}} = \left(\underline{\underline{a}}_{(AB)}^T \underline{\underline{c}}_{\phi A} \right)^T \bar{\mathbf{B}} \end{aligned} \quad (12b)$$

proving that

$$\underline{\underline{c}}_{\phi B} = \underline{\underline{a}}_{(AB)}^T \underline{\underline{c}}_{\phi A} \quad (12c)$$

ie that the representation coefficients in the 2 bases are related by the transpose of the passage-matrix relating the basis-elements. We may therefore write, using the passage-matrices $\underline{\underline{a}}_{(HT)} \in \mathbb{R}^{8 \times 8}$ (Tab. 1), $\underline{\underline{a}}_{(TH)} \in \mathbb{R}^{8 \times 8}$ (Tab. 2), $\underline{\underline{a}}_{(JT)} \in \mathbb{R}^{8 \times 8}$ (Tab. 3), $\underline{\underline{a}}_{(FT)} \in \mathbb{R}^{8 \times 8}$ (Tab. 4), $\underline{\underline{a}}_{(TF)} \in \mathbb{R}^{8 \times 8}$ (Tab. 5), and $\underline{\underline{a}}_{(TJ)} \in \mathbb{R}^{8 \times 8}$ (Tab. 6), expressing the basis-elements (4,5,8,9) of any of the bases as a linear combination of the basis-elements of another basis,

$$\begin{aligned} \frac{\phi_{ij}}{\rho \varepsilon} &= \sum_{n=1}^8 c_{\phi T_n} \bar{T}_{(n)ij} = \sum_{n=1}^8 c_{\phi H_n} \bar{H}_{(n)ij} \\ &= \sum_{n=1}^8 c_{\phi J_n} \bar{J}_{(n)ij} = \sum_{n=1}^8 c_{\phi F_n} \bar{F}_{(n)ij} \end{aligned} \quad (13a)$$

$$\underline{\underline{c}}_{\phi T} = \underline{\underline{a}}_{(HT)}^T \underline{\underline{c}}_{\phi H} = \underline{\underline{a}}_{(FT)}^T \underline{\underline{c}}_{\phi F} = \underline{\underline{a}}_{(JT)}^T \underline{\underline{c}}_{\phi J} \quad (13b)$$

$$\underline{\underline{c}}_{\phi H} = \underline{\underline{a}}_{(TH)}^T \underline{\underline{c}}_{\phi T} \quad (13c)$$

$$\underline{\underline{c}}_{\phi J} = \underline{\underline{a}}_{(TJ)}^T \underline{\underline{c}}_{\phi T} \quad (13d)$$

$$\underline{\underline{c}}_{\phi F} = \underline{\underline{a}}_{(TF)}^T \underline{\underline{c}}_{\phi T} \quad (13e)$$

6 Isotropic limit

Inspection of the matrices relating different bases (Tabs. 1–6) indicates that the model-coefficient (13a) of \bar{S}_{ij} changes depending on the basis used, and that in the symmetric bases, $\mathfrak{B}[\mathbf{b}, \bar{\mathbf{P}}, \bar{\mathbf{D}}]$ (8) and $\mathfrak{B}[\mathbf{b}, \bar{\mathbf{P}}_{\bar{S}}, \bar{\mathbf{P}}_{\bar{\Omega}}]$ (9), \bar{S}_{ij} can be represented as a linear combination of the elements of the basis (11), so that the Rotta-Crow² constraint [8, 9, 7, 12] can be easily represented by a linear relation between the model coefficients $\underline{\underline{c}}_{\phi F}$ (11a, 13b) or $\underline{\underline{c}}_{\phi J}$ (11b, 13b). The Rotta-Crow constraint [8, 9, 7, 12] requires that, at the limit of isotropic turbulence, the model for ϕ_{ij} should recover the analytical solution $\lim_{\mathbf{b} \rightarrow 0} \phi_{ij} = \frac{8}{10} k \bar{S}_{ij} = \frac{6}{10} \frac{4}{3} k \bar{S}_{ij}$. From the representation (11a) we readily have $\lim_{\mathbf{b} \rightarrow 0} \rho k \bar{S}_{ij} = -\frac{3}{4} P_{\bar{S}_{ij}}$. This suggests that an alternative expression of the Rotta-Crow constraint [8, 9, 7, 12] is

$$\lim_{\mathbf{b} \rightarrow 0} \phi_{ij} \stackrel{[1, 26, 8]}{=} \frac{6}{10} \frac{4}{3} k \bar{S}_{ij} \stackrel{(11a)}{=} -\frac{6}{10} P_{\bar{S}_{ij}} \quad (14)$$

The practical conclusion from (14) is that the Rotta-Crow constraint [8, 9, 7, 12] can be simply included in a closure expressed in $\mathfrak{B}[\mathbf{b}, \bar{\mathbf{P}}_{\bar{S}}, \bar{\mathbf{P}}_{\bar{\Omega}}]$ (9) as a constraint on the model-coefficient of $P_{\bar{S}_{ij}}^{(\text{dev})}$. The implication of this analysis goes beyond the idea of implicit satisfaction [9] of the Rotta-Crow constraint, as in the isotropization-of-production model [8], suggesting that representation bases built using $[\bar{\mathbf{P}}, \bar{\mathbf{D}}]$ or $[\bar{\mathbf{P}}_{\bar{S}}, \bar{\mathbf{P}}_{\bar{\Omega}}]$ need not contain explicitly \bar{S}_{ij} .

7 Discussion

The basic closure for ϕ_{ij} (slow return-to-isotropy, RI [1], and rapid isotropisation-of-production, IP [11]) uses the tensors b_{ij} (2c) and P_{ij} (2f), which have a clear physical significance. It was shown in the present work that the elements of the classic [6] representation basis $\mathfrak{B}[\mathbf{b}, \bar{\mathbf{S}}, \bar{\mathbf{\Omega}}]$ (4) can be represented in terms of production by strain $P_{\bar{S}_{ij}}$ (2d) or by rotation $P_{\bar{\Omega}_{ij}}$ (2e), as can be interpreted models [8, 9, 7] using their difference $D_{ij} = P_{\bar{S}_{ij}} - P_{\bar{\Omega}_{ij}}$ (2g). The widely known LRR [9] and SSG [4] models correspond to different weightings of the influence of $P_{\bar{S}_{ij}}$ (2d) and $P_{\bar{\Omega}_{ij}}$ (2e), the basic IP model [11] equally weights both, and the recent DY [12] proposal completely weights out $P_{\bar{\Omega}_{ij}}$ (2e).

To highlight this remark, consider the *a priori* term-by-term analysis of different SMCs, using DNS data [27] for fully

²Launder et al. [9] attribute to Crow [26, (3.6), p. 7] the behaviour of the rapid part of redistribution at the limit of isotropic turbulence, and this has been since repeated by several authors [7, 12]. Notice however [8] that the same constraint had also been given by Rotta [1, p. 558].

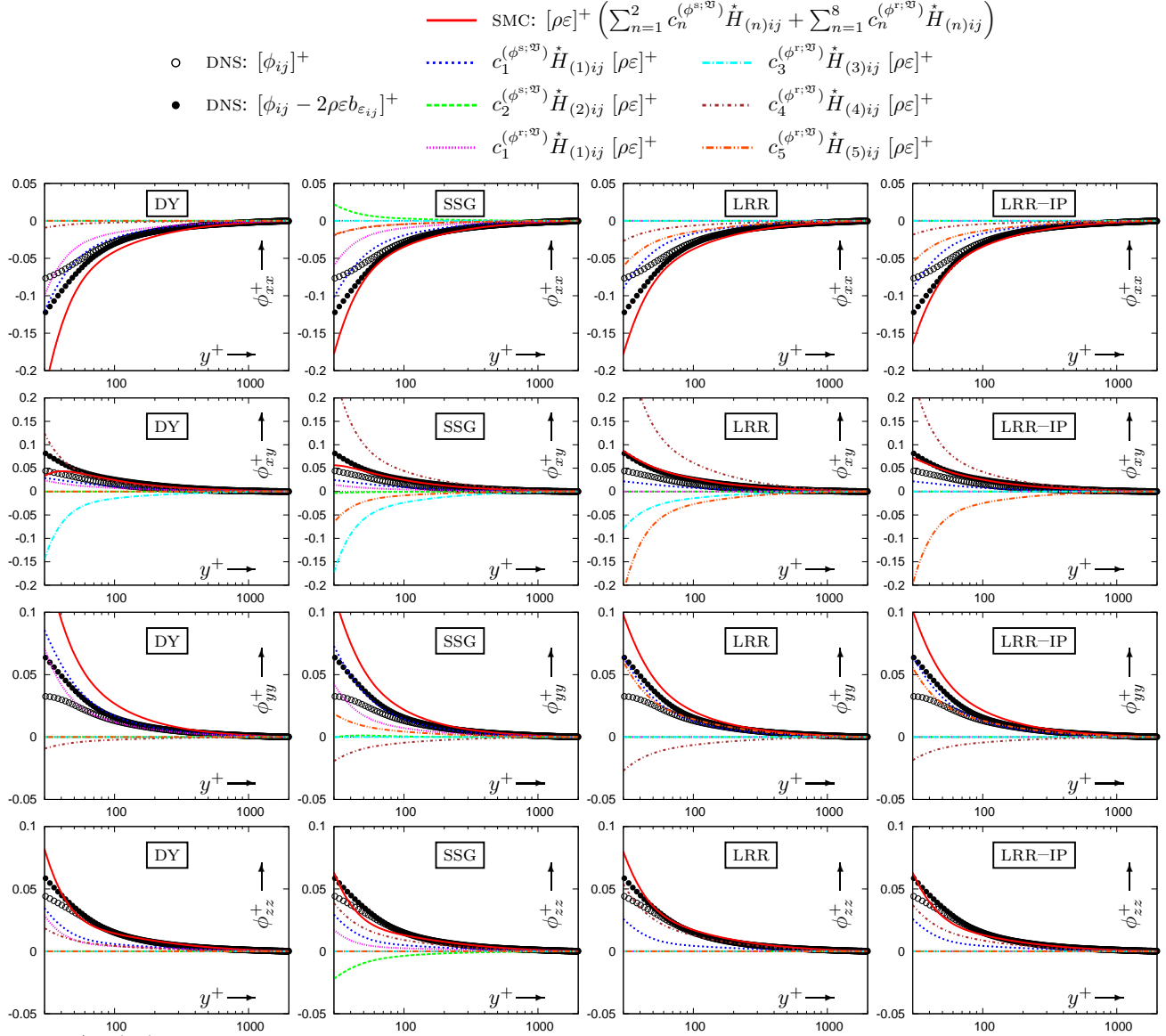


Fig. 2. *A priori* term-by-term analysis of SMCs (LRR [9], LRR-IP [9], SSG [4], DY [12]) for the components of the redistribution tensor ϕ_{ij} , eventually augmented by the anisotropy of dissipation ($\phi_{ij} - 2\rho\varepsilon b_{\varepsilon_{ij}} := \phi_{ij} - \rho(\varepsilon_{ij}^{(\mu)} - \frac{2}{3}\varepsilon)$), with respect to DNS data [27] for the log and outer regions ($y^+ \geq 30$) of fully developed incompressible plane channel flow at friction-Reynolds-number $Re_{\tau_w} = 2003$ [27].

developed incompressible plane channel flow, where the contribution of each basis-element to the closure of ϕ_{ij} is considered separately (Fig. 2). In this initial comparison, the models were projected in the basis $\mathfrak{B}_{(P)}[\mathbf{b}, \hat{\mathbf{S}}, \hat{\mathbf{P}}_{\hat{\Omega}}; \hat{\mathbf{P}}_{\hat{\mathcal{S}}}]$ (5), which gives the most compact representation, because 3 of the 4 models contain explicitly $\hat{\mathcal{S}}_{ij}$ [9, 4, 12]. Their comparison in the symmetric basis $\mathfrak{B}[\mathbf{b}, \hat{\mathbf{P}}_{\hat{\mathcal{S}}}, \hat{\mathbf{P}}_{\hat{\Omega}}]$ (9), will be the subject of future work. The contribution of $b_{ij} \stackrel{(5a)}{=} \hat{H}_{(1)ij}$ to the models was separated [6] to a slow part ($\rho\varepsilon c_1^{(\phi^{s;\mathfrak{U}})} \hat{H}_{(1)ij}$; Fig. 2) and a rapid part ($\rho\varepsilon c_1^{(\phi^{r;\mathfrak{U}})} \hat{H}_{(1)ij}$; Fig. 2) whose coefficient $c_1^{(\phi^{r;\mathfrak{U}})}$ is proportional to P_k (6b), *ie* dependent on mean-velocity gradients (the coefficient of this term is $\neq 0$ in the SSG [4] and the DY [12] closures). For the weakly inhomogeneous (the strongly inhomogeneous near-wall, $y^+ \leq 30$, region was not

plotted; Fig. 2) pure shear flow studied, the four different SMCs (LRR [9], LRR-IP [9], SSG [4], DY [12]) yield similar global results, clearly indicating that they all include the anisotropy of dissipation (Fig. 2). All of the closures use a very similar coefficient for the slow b_{ij} term ($\rho\varepsilon c_1^{(\phi^{s;\mathfrak{U}})} \hat{H}_{(1)ij}$; Fig. 2), which is the basic RI model [1]. Only the SSG [4] closure contains a nonlinear slow term ($\rho\varepsilon c_2^{(\phi^{s;\mathfrak{U}})} \hat{H}_{(2)ij}$), which reduces redistribution by increasing ϕ_{xx} (in the present case $\phi_{xx} < 0$) and reducing ϕ_{zz} (Fig. 2).

On the other hand, the different closures yield similar global predictions for the rapid part of redistribution, but with different weights on each of the 4 basis-elements with rapid coefficients $\neq 0$ (Fig. 2). The contributions of $P_{\hat{\Omega}_{ij}}$ ($\rho\varepsilon c_5^{(\phi^{r;\mathfrak{U}})} \hat{H}_{(5)ij}$) and $P_{\hat{\mathcal{S}}_{ij}}^{(\text{dev})}$ ($\rho\varepsilon c_4^{(\phi^{r;\mathfrak{U}})} \hat{H}_{(4)ij}$) to the LRR and LRR-IP closures [9] are quite similar (Fig. 2), the con-

tribution of $\bar{S}_{xy} \neq 0$ ($\rho \epsilon c_3^{(\phi; \tau; \Omega)} \dot{H}_{(3)ij}$) to the LRR [9] closure for ϕ_{xy} along with the contribution of $P_{\bar{S}_{xy}}^{(\text{dev})}$ yielding roughly the contribution of $P_{\bar{S}_{xy}}^{(\text{dev})}$ to the LRR–IP [9] closure, for which $c_3^{(\phi; \tau; \Omega)} = 0$. For the SSG [4] closure the contribution of $P_{\bar{\Omega}_{ij}}^{(\phi; \tau; \Omega)}$ ($\rho \epsilon c_5^{(\phi; \tau; \Omega)} \dot{H}_{(5)ij}$) is roughly $\frac{1}{3}$ compared to the LRR and LRR–IP closures [9], while the contribution of $P_{\bar{S}_{ij}}^{(\text{dev})}$ ($\rho \epsilon c_4^{(\phi; \tau; \Omega)} \dot{H}_{(4)ij}$) is roughly the same (Fig. 2). The SSG [4] closure compensates the reduced contribution of $P_{\bar{\Omega}_{ij}}$ by the rapid contribution of b_{ij} ($\rho \epsilon c_1^{(\phi; \tau; \Omega)} \dot{H}_{(1)ij}$), to yield a global prediction for the rapid part of redistribution similar to the LRR and LRR–IP closures [9] (Fig. 2). Nonetheless this is a very different modelling choice, since the SSG [4] coefficient of the rapid contribution of b_{ij} is proportional to $P_k \stackrel{(6b)}{=} \frac{1}{2} \mathbf{I}_P$, and hence is related to a mechanism associated with strain-production $P_{\bar{S}_{ij}}$ and not rotation-production $P_{\bar{\Omega}_{ij}}$, implying that the SSG closure weights more the former than the latter, compared to the LRR and LRR–IP closures [9]. Of course this is further amplified in the DY [12] closure (Fig. 2), which completely weights out the contribution of $P_{\bar{\Omega}_{ij}}$ ($\rho \epsilon c_5^{(\phi; \tau; \Omega)} \dot{H}_{(5)ij}$), further increasing the rapid contribution of b_{ij} ($\rho \epsilon c_1^{(\phi; \tau; \Omega)} \dot{H}_{(1)ij}$).

Establishing which is the best model is largely beyond the scope of the present note, requiring systematic comparison for a wide variety of flows. The purpose of the present comparison is, on the contrary, to highlight the usefulness of element-by-element comparison of different models.

8 Extensions

To represent ϕ_{ij} in strongly inhomogeneous cases, such as wall turbulence, let us assume, as do most models for near-wall turbulence [21], that we can identify (either geometrically or through gradients of local turbulence quantities) a unit vector pointing in the dominant direction of turbulence inhomogeneity, \vec{e}_η ($|\vec{e}_\eta| = 1$). By adding the deviatoric projection of the tensor product $\vec{e}_\eta \otimes \vec{e}_\eta$, viz $\boldsymbol{\eta} := \vec{e}_\eta \otimes \vec{e}_\eta - \frac{1}{3} \mathbf{I}_3$, to the 3 basis-generators we obtain (under the condition of linearity in mean-velocity gradients and the fact that all powers of $\boldsymbol{\eta}$ can be represented as a linear combination of $\boldsymbol{\eta}$ and \mathbf{I}_3 , because by straightforward computation $\boldsymbol{\eta}^2 = \frac{1}{3} \boldsymbol{\eta} + \frac{2}{9} \mathbf{I}_3$) 14 more basis-elements [15], which form, along with the 8 elements of the quasi-homogeneous basis, a 22-element representation basis for ϕ_{ij} in wall turbulence. As an initial application of these ideas, the 22-element basis obtained by combining $\boldsymbol{\eta}$ with the 3 generators of $\mathfrak{B}_{(P)}[\mathbf{b}, \dot{\mathbf{S}}, \dot{\mathbf{P}}_{\bar{\Omega}}; \dot{\mathbf{P}}_{\bar{S}}] := \{\dot{\mathbf{H}}_{(n)}, n \in \{1, \dots, 8\}\}$ (5) has been used [15] to represent and compare all known families of single-point SMCs in a common basis.

As usual [4, 6] the extension of closures for ϕ_{ij} to a reference frame rotating with angular velocity $\vec{\Omega}_{\text{RF}}$ is obtained by replacing everywhere $\bar{\Omega}_{ij}$ by the intrinsic (absolute) mean rotation-rate $\bar{W}_{ij} := \bar{\Omega}_{ij} - \epsilon_{ijk} \Omega_{\text{RF}k}$. The extension

of incompressible closures to compressible flows is obtained by replacing the mean strain-rate by its deviatoric projection $\bar{S}_{ij}^{(\text{dev})} := \bar{S}_{ij} - \frac{1}{3} \bar{S}_{\ell\ell} \delta_{ij}$ everywhere, including in the definition (1) of ϕ_{ij} [21].

9 Conclusions

By splitting the Reynolds-stress production-tensor P_{ij} (2f) into strain-production $P_{\bar{S}_{ij}}$ (2d) and rotation-production $P_{\bar{\Omega}_{ij}}$ (2e), $P_{ij} \stackrel{(2f)}{=} P_{\bar{S}_{ij}} + P_{\bar{\Omega}_{ij}}$, the physical interpretation of the tensor D_{ij} (2g), present in the original expression of many closures both early [2, 8, 9] and more recent [7, 23], is obtained as the difference between the two mechanisms of production, $D_{ij} \stackrel{(2g)}{=} P_{\bar{S}_{ij}} - P_{\bar{\Omega}_{ij}}$.

The classical representation basis [6] $\mathfrak{B}[\mathbf{b}, \dot{\mathbf{S}}, \dot{\boldsymbol{\Omega}}]$ (4) of closures for pressure-strain redistribution ϕ_{ij} , can be replaced by equivalent representation bases, and model coefficients in different bases can be calculated in a systematic way, using passage-matrices (§5). Such representation bases, eg $\mathfrak{B}[\mathbf{b}, \dot{\mathbf{P}}_{\bar{S}}, \dot{\mathbf{P}}_{\bar{\Omega}}]$ (9), are generated exclusively by tensors appearing in the transport equations for the Reynolds-stresses, in which the rate-of-strain tensor \bar{S}_{ij} (2a) is weighted by the anisotropy tensor b_{ij} (2c). In this basis, \bar{S}_{ij} can be explicitly represented, using the Cayley-Hamilton theorem and its extensions [17], as a linear combination of b_{ij} , $P_{\bar{S}_{ij}}$ and their powers/products which are linear in mean-velocity gradients (11a), and the Rotta-Crow constraint [1, 26] appears as a limiting constraint on the model-coefficient of $P_{\bar{S}_{ij}}$. The practical implication of these results is that we can use the bases $\mathfrak{B}[\mathbf{b}, \dot{\mathbf{P}}_{\bar{S}}, \dot{\mathbf{P}}_{\bar{\Omega}}]$ (9) or $\mathfrak{B}[\mathbf{b}, \dot{\mathbf{P}}, \dot{\mathbf{D}}]$ (8) to compare different models, including those constructed in the classical basis $\mathfrak{B}[\mathbf{b}, \dot{\mathbf{S}}, \dot{\boldsymbol{\Omega}}]$ (4), but also to construct general models for ϕ_{ij} which do not contain explicitly the numerically stiff term \bar{S}_{ij} [28].

An initial term-by-term decomposition on the basis-elements of different models (Fig. 2) illustrates how substantially different weighting of various basis-tensors can lead to very similar global results in weakly inhomogeneous pure shear flow. The detailed element-by-element analysis of different models for various types of flows, using the bases $\mathfrak{B}[\mathbf{b}, \dot{\mathbf{P}}_{\bar{S}}, \dot{\mathbf{P}}_{\bar{\Omega}}]$ (9) or $\mathfrak{B}[\mathbf{b}, \dot{\mathbf{P}}, \dot{\mathbf{D}}]$ (8), will be the subject of future work.

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